THE REMAINDER IN TAUBERIAN THEOREMS

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Papers summarized in the dissertation

I. LYTTKENS, S., The remainder in Tauberian theorems. *Arkiv för matematik, 2* (1954), 575–588.

II. LYTTKENS, S., The remainder in Tauberian theorems II. *Arkiv för matematik, 3* (1956), 315–349.

THE REMAINDER IN TAUBERIAN THEOREMS

Most classical theorems in the complex of problems known as Tauberian originate from pioneer work done by Hardy and Littlewood, who designated them after the first theorem of this kind established by Tauber. Roughly speaking, a Tauberian theorem asserts that if an average of a function, $\Phi(x)$, behaves regularly as x tends toward a limit then $\Phi(x)$ behaves regularly at that limit. This, of course, is only true provided that some supplementary, so-called Tauberian conditions are imposed on $\Phi(x)$, which are mainly one-sided restrictions on the oscillations of $\Phi(x)$. A special Tauberian theorem deals with a specific average or summation. Before the work of Wiener, all Tauberian theorems were "special" and methods of proof varied, being adapted to a particular problem; for example, Hardy and Littlewood applied a repeated differentiation technique to problems concerning Abel and Cesàro summations. In 1930 Karamata published his ingenious method of polynomial approximation to prove a classical theorem of Hardy and Littlewood, a method which has proved its usefulness for various other Tauberian problems. Wiener's general theory, presented in 1932, systematized and extended this complex of problems by making use of Fourier transforms and closure properties. Wiener's work was later generalized and simplified by other writers, in particular by Pitt. These writers proved that most Tauberian theorems can be reduced to the following general form:

If $K(x) \in L$ and $k(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} K(x)\, dx \neq 0$, then the relation

$$\Phi * K(x) = o(1) \qquad \text{as } x \to \infty$$

implies that

$$\Phi(x) = o(1) \qquad \text{as } x \to \infty$$

for every bounded function, $\Phi(x)$, which satisfies certain Tauberian conditions.

Later, besides this classical type, Tauberian theorems have appeared which we shall call *remainder Tauberian theorems*. They represent a refinement of the ordinary Tauberian theorems and can be described as follows: Reduced to the above form, a remainder theorem investigates the order of magnitude of $\Phi(x)$ as $x \to \infty$, provided that the order of magnitude of $\Phi * K(x)$ as $x \to \infty$ is known.

The first general theorems of this kind were stated by Beurling in [2], and the conditions imposed on $K(x)$ can be considered as an extension of the Wiener condition, $k(\xi) \neq 0$. One of the first special theorems of this kind was a theorem concerning Stieltjes transforms stated by Carleman in [3]. In recent years a number of other remainder theorems have been added. Most of these are special Tauberian theorems concerning Laplace and Stieltjes transforms, solved by refinements of Karamata's method, where no use is made of transform properties of kernels involved.

Tauberian problems with exponentially vanishing remainders are examined in paper [I] where modifications of the above theorems by Beurling are proved. Conditions are stated under which the relation

$$\Phi * K(x) = O(e^{-\gamma x}) \qquad \text{as } x \to \infty$$

implies that

$$\Phi(x) = O(e^{-\theta \gamma x}) \text{ as } x \to \infty$$

for every bounded function, $\Phi(x)$, which satisfies certain Tauberian conditions, $\theta = \theta_K$ being a positive constant. (In fact, our theorems are stated in a slightly more general form with $K(x)\,dx$ replaced by $dF(x)$, $F(x)$ being a function of bounded variation.) In chapter 1 of (II) corresponding results are derived for a more general type of remainder. Also it is proved that some conditions postulated in certain theorems in [I] can be relaxed. The method of proof follows Wiener in making use of properties of the Fourier transform, $k(\xi)$, of $K(x)$. Conditions imposed on $k(\xi)$ consist, first, of the analyticity of $1/k(\xi)$ in a certain strip below the real axis. Second, for $\zeta = \xi + i\,\eta$, the function $1/k(\zeta)$ is supposed to be dominated by a power of $|\xi|$ when $|\xi|$ tends to infinity in this strip. The Tauberian conditions are slightly broadened generalisations of the following condition: $\Phi(x) + Cx$ is non-decreasing for $x > x_0$ and some constant C.

The major part of the paper [II] is an investigation of the necessity of the conditions postulated in the above-mentioned theorems. It is proved, for a large class of kernels, that the analyticity of $1/k(\xi)$ in a certain strip below the real axis is a necessary condition. In conclusion, an example is presented proving that the value of the constant θ, derived in Theorems 3 and 4 in [I], is essential.

It is natural to expect that the stronger a method of summation, the weaker are the results which can be obtained by corresponding remainder Tauberian theorems. The theorems in [I] and in chapter 1 of [II] can be applied, for example, to Tauberian problems concerning Riesz and Cesàro summability. They are not applicable to Tauberian problems for the Abel summability or for Laplace transforms, because of the above-mentioned restrictions on the order of magnitude of $1/k(\xi)$. And it is well known that results of the type cited above cannot be obtained if we let $K(x)$ denote the kernel corresponding to an Abel summation. However, it seems likely that similar methods can be applied to remainder problems of this type.

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